Non homogeneous differential equations

Let us consider a second order linear non homogeneous DE with constant co-efficient

$$y'' + ay' + by = f(x) \quad \text{---(1)}$$

or the second order non homogeneous Euler Cuchy equation

$$x^2y'' + axy' + by = f(x) \quad \text{---(2)}$$

The general solution of equation (1) or (2) is given by

$$\boxed{y = y_c + y_p}$$

where $y_c \rightarrow$ Complementary function

$y_p \rightarrow$ Particular Integral.

Method of calculation of y_c

STEP:-I Write down the homogeneous form of equation (1) or (2) which one is given i.e

$$y'' + ay' + by = 0$$

or $x^2y'' + axy' + by = 0$

STEP:-II

Find out the auxilary or the characterstics equation.

STEP:-III

Find out the roots of the auxilary equation.

Scanned with CamScanner

STEP:-IV

Apply the formula from the given table to get y_c

Table

ROOTS	$y'' + ay' + by = 0$	$x^2y'' + axy' + by = 0$
Real & Unequal	λ_1, λ_2 $y_c = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$	m_1, m_2 $y_c = c_1 x^{m_1} + c_2 x^{m_2}$
Real & equal	λ, λ $y_c = (c_1 + c_2 x) e^{\lambda x}$	m, m $y_c = (c_1 + c_2 \ln x) x^m$
Complex & Conjugate	$\alpha \pm i\beta$ $y_c = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$	$\alpha \pm i\beta$ $y_c = x^{\alpha}(c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x))$

Methods for finding out y_p

There are two methods for finding out the particular integral y_p

1. Method of undetermined co-efficient
2. Method of variation of parameter.

These two methods we will discuss subsiquently

Scanned with CamScanner

PAGE-103 (E. KREYSZIG) (8TH EDITION)

NO-1-TO-8

Verify that y_p is a solution of the given differential equation and find a general solution.

1. $y'' - y = 8e^{-3x}$ ——— (1) $\quad y_p = e^{-3x}$.

Ans:- we have,

$$y_p = e^{-3x}$$

$$y_p' = -3e^{-3x}$$

$$y_p'' = 9e^{-3x}$$

LHS:-

$$y'' - y$$

$$= 9e^{-3x} - e^{-3x}$$

$$= 8e^{-3x} = \text{RHS}$$

So, $y_p = e^{-3x}$ is a solution of equation (1)

The homogeneous form of equation (1) is

$$y'' - y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} - y = 0$$

$$\Rightarrow (D^2 - 1)y = 0 \qquad \because D = \frac{d}{dx}$$

The auxilary equation is

$$\lambda^2 - 1 = 0$$

$$\Rightarrow \lambda = \pm 1$$

Thus, $y_c = c_1 e^{-x} + c_2 e^{x}$

The general solution is

$$y = y_c + y_p$$

$$\Rightarrow y = c_1 e^{-x} + c_2 e^{x} + e^{-3x}.$$

Scanned with CamScanner

3. $y''+3y'+2y = 4x^2$ ——— (1) $\quad y_p = 2x^2-6x+7$

Ans:- we have,

$$y_p = 2x^2-6x+7$$

$$y_p' = 4x-6$$

$$y_p'' = 4$$

LHS:-

$$y''+3y'+2y$$

$$= 2x^2-6x+7+12x-18+8$$

$$= 4+12x-18+4x^2-12x+14$$

$$= 4x^2 = \text{RHS}$$

Thus $y_p = 2x^2-6x+7$ is a solution of equation (1)

The homogeneous form of equation (1) is

$$y''+3y'+2y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y = 0$$

$$\Rightarrow (D^2+3D+2)\,y = 0 \qquad (\because D = \frac{d}{dx})$$

The auxilary equation is

$$\lambda^2+3\lambda+2 = 0$$

$$\Rightarrow (\lambda+1)(\lambda+2) = 0$$

$$\Rightarrow \lambda = -1, -2$$

So, $y_c = c_1e^{-x}+c_2e^{-2x}$

Hence the general solution is

$$y = y_c+y_p$$

$$\Rightarrow y = c_1e^{-x}+c_2e^{-2x}+2x^2-6x+7.$$

Scanned with CamScanner

9-TO-15

Verify y_p is a solution of the given equation. Solve the initial value problem.

9. $y'' + y = 2x$ ——— (1) $\quad y(0) = -1,\ y'(0) = -8,\ y_p = 2x$

Ans:- We have, $y_p = 2x$

$$y_p' = 2$$

$$y_p'' = 0$$

LHS:- $y'' + y$

$= 0 + 2x = 2x = RHS$

Thus $y_p = 2x$ is a solution of equation (1)

The homogeneous form of equation (1) is

$$y'' + y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + y = 0$$

$$\Rightarrow (D^2 + 1)y = 0 \qquad \left(\because D = \frac{d}{dx}\right)$$

The auxiliary equation is

$$\lambda^2 + 1 = 0$$

$$\Rightarrow \lambda = \pm i$$

Thus. $y_c = e^{0x}(C_1 \cos x + C_2 \sin x)$

$$\Rightarrow y_c = C_1 \cos x + C_2 \sin x .$$

The general solution is

$$y = y_c + y_p$$

$$\Rightarrow y = C_1 \cos x + C_2 \sin x + 2x \quad \text{——— (2)}$$

$$y' = -C_1 \sin x + C_2 \cos x + 2 \quad \text{——— (3)}$$

$y(0) = -1$ i.e $y = -1$ when $x = 0$ from (2)

$$-1 = C_1 \cos 0 + C_2 \sin 0 + 2 \cdot 0$$

$$\Rightarrow -1 = C_1 \quad \text{——— (3)}$$

Scanned with CamScanner

$y'(0) = -8$ i.e. $y' = -8$ when $x = 0$ from (3)

$$-8 = -C_1 \sin 0 + C_2 \cos 0 + 2$$

$$\Rightarrow -8 = C_2 + 2$$

$$\Rightarrow C_2 = -10 \quad \text{——— (4)}$$

Putting the value of C_1 and C_2 in equation (2) the solution is

$$y = -\cos x - 10 \sin x + 2x.$$

13. $(x^2D^2 - 3xD + 3)y = 3\ln x - 4$ ——— (1)

$y(1) = 0$, $y'(1) = 1$, $y_p = \ln x$

Ans:- we have, $y_p = \ln x$

$$Dy_p = y_p' = \frac{1}{x}$$

$$D^2y_p = y_p'' = -\frac{1}{x^2}$$

LH.S:.

$$(x^2D^2 - 3xD + 3)y$$

$$= x^2\left(-\frac{1}{x^2}\right) - 3x\left(\frac{1}{x}\right) + 3\ln x$$

$$= -1 - 3 + 3\ln x$$

$$= 3\ln x - 4 = \text{RHS}$$

Thus, $y_p = \ln x$ is a solution of equation (1)

The homogeneous form of equation (1) is

$$(x^2D^2 - 3xD + 3)y = 0 \quad \text{——— (2)}$$

Put $y = x^m$

$$Dy = y' = mx^{m-1}$$

$$D^2y = y'' = m(m-1)x^{m-2}$$

Putting these values in equation (2)

$$[m(m-1) - 3m + 3]x^m.$$

The auxiliary equation is

Scanned with CamScanner

$$m(m-1) - 3m + 3 = 0$$

$$\Rightarrow m^2 - 4m + 3 = 0$$

$$\Rightarrow (m-1)(m-3) = 0$$

$$\Rightarrow m = 1, 3.$$

Thus, $y_c = c_1 x^1 + c_2 x^3$

The general solution is

$$y = y_c + y_p$$

$$\Rightarrow y = c_1 x + c_2 x^3 + \ln x \quad \text{——— (3)}$$

$$y' = c_1 + 3c_2 x^2 + \frac{1}{x} \quad \text{——— (4)}$$

$y(1) = 0$ i.e $y = 0$ when $x = 1$ from equation (3)

$$0 = c_1 + c_2 + \ln 1$$

$$\Rightarrow c_1 + c_2 = 0 \quad \text{——— (5)}$$

$y'(1) = 1$ i.e $y' = 1$ when $x = 1$ from equation (4)

$$1 = c_1 + 3c_2 + 1$$

$$\Rightarrow c_1 + 3c_2 = 0 \quad \text{——— (6)}$$

Solving equation (5) and (6)

$$c_1 = 0, \; c_2 = 0$$

Putting the value of c_1 and c_2 in equation (2) the solution is

$$y = 0x + 0x^3 + \ln x$$

$$\Rightarrow y = \ln x.$$

Scanned with CamScanner